

WEAKLY AND ALMOST WEAKLY STABLE C_0 -SEMIGROUPS

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ABSTRACT. In this article we survey results concerning asymptotic properties of C_0 -semigroups on Banach spaces with respect to the weak operator topology. The property “no eigenvalues of the generator on the imaginary axis” is equivalent to weak stability for most time values; a phenomenon called “almost weak stability”. Further, sufficient conditions implying the actual weak stability of a C_0 -semigroup are also given. The individual stability versions are considered as well, using both boundedness of the local resolvent and integrability conditions for the resolvent operator. By a number of examples we explain weak and almost weak stability and illustrate the fundamental difference between them. Where adequate, connections to strong stability are observed, while many historical and bibliographical remarks position the material in the literature. We conclude the paper with some open questions and comments.

1. INTRODUCTION

Strongly continuous semigroups on Banach spaces (C_0 -semigroups for short) provide a very efficient and elegant tool for the treatment of concrete and abstract Cauchy problems (see Engel, Nagel [17]). They not only yield well-posedness results (through the classical Hille-Yosida theorem and its variants), but also allow a detailed description of important qualitative properties of the solutions of the Cauchy problem. In this context, it is important to describe the asymptotic behaviour of the solutions. In terms of the semigroup, this means the following.

Problem 1.1. Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup with generator A on a Banach space X . For each $x \in X$ describe the behaviour of $T(t)x$ as $t \rightarrow \infty$.

This question has been studied systematically in the monograph [46] by van Neerven, while Section V in Engel, Nagel [17] or Sections V and VI in Engel, Nagel [18] contain the major abstract results and some concrete applications.

For the following we assume X to be reflexive and $(T(t))_{t \geq 0}$ to be bounded. Then it follows from the Jacobs-Glicksberg-de Leeuw theory (see [17], Thm. V.2.8) that X is the direct sum of the two subspaces

$$\begin{aligned} X_r &:= \overline{\text{lin}}\{x \in D(A) : Ax = i\lambda x \text{ for some } \lambda \in \mathbb{R}\} \\ &= \overline{\text{lin}}\{x \in X : T(t)x = e^{i\lambda t}x \text{ for some } \lambda \in \mathbb{R}\}, \\ X_s &:= \{x \in X : 0 \text{ is a weak accumulation point of } \{T(t)x : t \geq 0\}\}. \end{aligned}$$

The restriction of $(T(t))_{t \geq 0}$ to X_r acts as a group and its behaviour is determined by the purely imaginary eigenvalues $i\lambda \in P\sigma(A)$ of the generator A (as usual $P\sigma(A)$ denotes the point spectrum of the operator A), or, equivalently, by the periodic orbits of $T(t)$. Therefore, for this part of $(T(t))_{t \geq 0}$ Problem 1.1 is solved in a reasonable way.

The fourth author has been supported by the Marie Curie Host Fellowship “Spectral theory for evolution equations”, contract number HPMT-CT-2001-00315.

2000 *Mathematics Subject Classification.* 47D06, 37A25.

Key words and phrases. C_0 -semigroups, weak operator topology, stability, mixing.

On the other hand, A has no eigenvectors with purely imaginary eigenvalue in X_s and, based on the above characterisation, one expects

$$\text{“} \lim_{t \rightarrow \infty} T(t)x = 0 \text{” for all } x \in X_s$$

in some sense. Therefore, Problem 1.1 reduces to the following.

Problem 1.2. Let $(T(t))_{t \geq 0}$ be a bounded C_0 -semigroup with generator A on a reflexive Banach space X . Find necessary and sufficient conditions for $(T(t))_{t \geq 0}$ to be *stable*, i.e., to satisfy

$$\lim_{t \rightarrow \infty} T(t)x = 0 \quad \text{for all } x \in X_s,$$

where the limit is taken in some appropriate topology.

The first result in this direction goes back to Lyapunov. In 1892 he gave the following characterisation for matrix semigroups.

Theorem 1.3. Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup with generator A on a finite dimensional Banach space X . The semigroup $(T(t))_{t \geq 0}$ is uniformly stable, i.e., $\lim_{t \rightarrow \infty} \|T(t)\| = 0$ if and only if

$$s(A) := \sup \{\operatorname{Re} \lambda : \lambda \in \sigma(A)\} < 0.$$

This characterisation does no longer hold on infinite dimensional spaces and for unbounded generators (see [17], Sect. IV.3 for counterexamples). Still, the situation concerning uniform stability is quite well understood and there are many results, using additional properties of the Banach space and/or the semigroup, on uniformly stable semigroups. We refer to Engel, Nagel [17], Sect. V.1.b, Arendt, Batty, Hieber, Neubrander [2], Sect. 5.2-5.3, and van Neerven [46], Ch. 3.

The *strong stability* of $(T(t))_{t \geq 0}$, i.e., the property that

$$\lim_{t \rightarrow \infty} \|T(t)x\| = 0 \quad \text{for all } x \in X,$$

is not so well understood and only in 1988 did Arendt, Batty [1], Lyubich, Vũ [40] obtain the following sufficient, but not necessary condition.

Theorem 1.4. Let $(T(t))_{t \geq 0}$ be a bounded C_0 -semigroup with generator A on a reflexive Banach space X with $P_\sigma(A) \cap i\mathbb{R} = \emptyset$. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t \geq 0}$ is strongly stable.

More results on strong stability with even a necessary and sufficient condition in the case of Hilbert spaces are due to Tomilov [59] and Chill, Tomilov [9, 10].

Surprisingly, compared to the above there are relatively few results on *weakly stable* semigroups $(T(t))_{t \geq 0}$, i.e., semigroups satisfying

$$\lim_{t \rightarrow \infty} \langle T(t)x, \varphi \rangle = 0 \quad \text{for all } x \in X \text{ and all } \varphi \in X'.$$

Note that, for example, isometric semigroups (in particular unitary groups) never converge strongly (except in the trivial case), so convergence with respect to the weak topology is the natural mode of convergence for such semigroups.

The absence of a theory for weak stability of semigroups is regrettable not just for pure mathematical reasons. In the setting of quantum theory, one thinks of X as the state space, while X' is the space of observables of some system. Therefore, the scalar valued function

$$t \mapsto \langle T(t)x, \varphi \rangle$$

gives the time evolution of a measuring process. Consequently, weak stability is the property of a system which can indeed be observed.

Also in ergodic theory, the concept of weak stability occurs naturally (under the name “strong mixing”). We quote here the following from Katok, Hasselblatt [32], p. 748.

“... *It [strong mixing] is, however, one of those notions, that is easy and natural to define but very difficult to study...*”

We will see later that a weaker concept (“almost weak stability”, see Section 2) is in fact relatively easy to characterise, whereas our knowledge about weak stability itself is still limited.

The aim of this paper is thus to survey the known results on weak asymptotic properties of C_0 -semigroups, and also to propose some ideas for further research.

We will start with the essentially easier notion of “*almost weak stability*”. By virtue of the Jacobs-Glicksberg-de Leeuw decomposition, for bounded semigroups on reflexive spaces, almost weak stability turns out to be equivalent to

the generator A has no purely imaginary eigenvalues.

In Section 2 we will give a series of equivalent properties justifying the term “almost weakly stable”, and give a brief account on the history of this and related notions. In particular, we mention *weakly almost periodic functions* to which we return in the concluding Section 6 as well.

In Section 3, we proceed to weak stability. In Hilbert spaces, one can decompose contraction semigroups into a weakly stable and a unitary part, which allows us to restrict the study to unitary groups. In general Banach spaces, we present sufficient conditions for convergence to 0 of orbits of C_0 -semigroups in terms of integrability of the resolvent of the generator. We also discuss how the convergence of the semigroups $(T(t))_{t \geq 0}$ and $(T(n))_{n \in \mathbb{N}}$ is related.

We devote Section 4 to examples and counterexamples to illustrate the notions of weak and almost weak stability. In particular, we show that in a certain sense only a minority of almost weakly stable semigroups are indeed weakly stable.

Section 5 is devoted to the property

$$\lim_{t \rightarrow \infty} \langle T(t)x, y \rangle = 0 \quad \text{for some given } x \text{ and } y,$$

the so-called *weak individual stability*. We investigate this phenomenon assuming the existence of a bounded local resolvent. Here the borderline between strong and weak individual stability is very narrow: in the presence of certain geometric properties of the underlying Banach space one even obtains strong stability. We also elaborate on this aspect by recalling several results from the literature.

The last Section 6 is intended to reveal connections to results which do not exactly fit into the line of this survey, but are connected to our topic and thus deserve a few words. We finally pose some open questions.

2. ALMOST WEAK STABILITY

Let us start by discussing almost weak stability of C_0 -semigroups, a concept which is close to weak stability but much easier to investigate.

Our main functional analytic tool will be (relative) compactness for the weak operator topology. We denote by $\mathcal{L}_\sigma(X)$ the space $\mathcal{L}(X)$ endowed with the weak operator topology and recall the following characterisation of relative compactness in $\mathcal{L}_\sigma(X)$.

Lemma 2.1 ([17], Lemma V.2.7). *For a set of operators $\mathcal{T} \subset \mathcal{L}(X)$, X a Banach space, the following assertions are equivalent.*

- (a) \mathcal{T} is relatively compact in $\mathcal{L}_\sigma(X)$.
- (b) $\{Tx : T \in \mathcal{T}\}$ is relatively weakly compact in X for all $x \in X$.
- (c) \mathcal{T} is bounded, and $\{Tx : T \in \mathcal{T}\}$ is relatively weakly compact in X for all x in some dense subset of X .

Let us give some examples of relatively weakly compact subsets of operators.

Example 2.2.

- (a) On a reflexive Banach space X any norm-bounded family $\mathcal{T} \subseteq \mathcal{L}(X)$ is relatively weakly compact.
- (b) Let $\mathcal{T} \subseteq \mathcal{L}(L^1(\mu))$ be a norm-bounded subset of positive operators on the Banach lattice $L^1(\mu)$, and suppose that $Tu \leq u$ for some μ -almost everywhere positive $u \in L^1(\mu)$ and every $T \in \mathcal{T}$. Then \mathcal{T} is relatively weakly compact since the order interval $[-u, u]$ is weakly compact and \mathcal{T} -invariant (see Schaefer [56], Thm. II.5.10 (f) and Prop. II.8.3).
- (c) Let S be a semi-topological semigroup, i.e., a (multiplicative) semigroup S which is a topological space such that the multiplication is separately continuous (see Engel, Nagel [17], Sec. V.2). Consider the space $C(S)$ of bounded, continuous (real- or complex-valued) functions over S . For $s \in S$ define the corresponding rotation operator $(L_s f)(t) := f(s \cdot t)$. A function $f \in C(S)$ is said to be *weakly almost periodic* if the set $\{L_s f : s \in S\}$ is relatively weakly compact in $C(S)$, see Berglund, Junghenn, Milnes [7], Def. 4.2.1 (cf also Section 6.1). The set of weakly almost periodic functions is denoted by $WAP(S)$. If S is a *compact* semi-topological semigroup, then $C(S) = WAP(S)$ holds, see [7], Cor. 4.2.9. This means that for a compact semi-topological semigroup S the set $\{L_s : s \in S\}$ is always relatively weakly compact. We come back to this example in Example 4.3 and in the proof of Theorem 2.5.

We now turn our attention to C_0 -semigroups (see Engel, Nagel [17] for the general theory).

Definition 2.3. A C_0 -semigroup $(T(t))_{t \geq 0}$ on a Banach space X is called *relatively weakly compact* if the set $\mathcal{T} := \{T(t) : t \geq 0\}$ satisfies one of the equivalent conditions in Lemma 2.1.

In the following we will concentrate only on relatively weakly compact semigroups. We note that every weakly stable semigroup has this property, therefore this is not a strong restriction with respect to our aims.

The following property of relatively weakly compact semigroups will be used in the sequel without explicit reference, see Engel, Nagel [17], Sec. V.4.

Proposition 2.4. *Let $(T(t))_{t \geq 0}$ be a relatively weakly compact semigroup on a Banach space X . Then it is mean ergodic, i.e., we have (weak and even strong) convergence of the Cesàro means*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t T(s)x \, ds = Px \quad \text{for all } x \in X,$$

where $P \in \mathcal{L}(X)$ is a projection onto $\text{Fix}(T) := \bigcap_{t \geq 0} \text{Fix}(T(t))$, the so-called ergodic projection.

Assume now $(T(t))_{t \geq 0}$ to be relatively weakly compact. By the decomposition theorem of Jacobs-Glicksberg-de Leeuw the Banach space X is a direct sum of the two invariant subspaces

$$\begin{aligned} X_r &= \overline{\text{lin}}\{x \in D(A) : Ax = i\lambda x \text{ for some } \lambda \in \mathbb{R}\}, \\ X_s &= \{x \in X : 0 \text{ is a weak accumulation point of } \{T(t)x : t \geq 0\}\}. \end{aligned}$$

(See Maak [41], Jacobs [27], Glicksberg, de Leeuw [20] and ultimately Krengel [33], Sec. 2.4 for detailed discussion and historical remarks and also Engel, Nagel [17], Thm. V.2.8).

So we see that the property “no eigenvalues of the generator on the imaginary axis” is equivalent to the fact that 0 is a weak accumulation point of the orbit $\{T(t)x : t \geq 0\}$ for every $x \in X$. The following theorem gives more detailed information about the asymptotic behaviour of the orbits in this case.

Theorem 2.5. *Let $(T(t))_{t \geq 0}$ be a relatively weakly compact C_0 -semigroup on a Banach space X with generator A . The following assertions are equivalent.*

- (i) $0 \in \overline{\{T(t)x : t \geq 0\}}^\sigma$ for every $x \in X$;
- (i') $0 \in \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma}$;
- (ii) For every $x \in X$ there exists a sequence $\{t_n\}_{n=1}^\infty$ with $t_n \rightarrow \infty$ such that $T(t_n)x \xrightarrow{\sigma} 0$;
- (iii) For every $x \in X$ there exists a set $M \subset \mathbb{R}_+$ with density 1 such that $T(t)x \xrightarrow{\sigma} 0$, as $t \in M, t \rightarrow \infty$;
- (iv) $\frac{1}{t} \int_0^t |\langle T(s)x, y \rangle| ds \xrightarrow[t \rightarrow \infty]{} 0$ for all $x \in X, y \in X'$;
- (v) $\lim_{a \rightarrow 0+} a \int_{-\infty}^\infty |\langle R(a+is, A)x, y \rangle|^2 ds = 0$ for all $x \in X, y \in X'$;
- (vi) $\lim_{a \rightarrow 0+} a R(a+is, A)x = 0$ for all $x \in X$ and $s \in \mathbb{R}$;
- (vii) $P\sigma(A) \cap i\mathbb{R} = \emptyset$, i.e., A has no purely imaginary eigenvalues.

If, in addition, X' is separable, then the conditions above are also equivalent to

- (ii*) There exists a sequence $\{t_n\}_{n=1}^\infty$ with $t_n \rightarrow \infty$ such that $T(t_n) \xrightarrow{\sigma} 0$;
- (iii*) There exists a set $M \subset \mathbb{R}_+$ with density 1 such that $T(t) \xrightarrow{\sigma} 0$, $t \in M$ and $t \rightarrow \infty$.

Recall that the (asymptotic) density of a measurable set $M \subset \mathbb{R}_+$ is

$$d(M) := \lim_{t \rightarrow \infty} \frac{1}{t} \lambda([0, t] \cap M),$$

whenever the limit exists (here λ is the Lebesgue measure on \mathbb{R}).

We will use the following classical lemma (for the proof see Petersen [48], Lemma 6.2).

Lemma 2.6. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous and bounded. The following assertions are equivalent.*

- (a) $\frac{1}{t} \int_0^t f(s) ds \rightarrow 0$ as $t \rightarrow \infty$;
- (b) There exists a set $M \subset \mathbb{R}_+$ with density 1 such that $f(t) \rightarrow 0$, $t \in M$ and $t \rightarrow \infty$.

Proof of Theorem 2.5.

The proof of the implication (i') \Rightarrow (i) is trivial. The implication (i) \Rightarrow (ii) holds since in Banach spaces weak compactness and weak sequential compactness coincide (see Eberlein-Šmulian theorem, e.g., Dunford, Schwartz [12], Thm. V.6.1).

If (vii) does not hold, then (ii) can not be true by the spectral mapping theorem for the point spectrum (see Engel, Nagel [17], Thm. V.3.7), hence (ii) \Rightarrow (vii).

The implication (vii) \Rightarrow (i') is the main consequence of the Jacobs-Glicksberg-de Leeuw theorem and follows from the construction in its proof, see Engel, Nagel [17], p. 313.

This proves the equivalences (i) \Leftrightarrow (i') \Leftrightarrow (ii) \Leftrightarrow (vii).

(vi) \Leftrightarrow (vii): Since the semigroup $(T(t))_{t \geq 0}$ is mean ergodic and bounded, the decomposition $X = \ker A \oplus \overline{\text{rg } A}$ holds (see [17], Lemma V.4.4). This implies by the mean ergodic theorem (see, e.g., Arendt, Batty, Hieber, Neubrander [2], Cor. 4.3.2) that the limit

$$Px := \lim_{a \rightarrow 0^+} aR(a, A)x$$

exists for all $x \in X$ with a projection P onto $\ker A$. Therefore, $0 \notin P\sigma(A)$ if and only if $P = 0$. Take now $s \in \mathbb{R}$. The semigroup $(e^{ist}T(t))_{t \geq 0}$ is also relatively weakly compact and hence mean ergodic. Repeating the argument for this semigroup we obtain (vi) \Leftrightarrow (vii).

(i') \Rightarrow (iii): Let $S := \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma} \subseteq \mathcal{L}(X)$ which is a compact semi-topological semigroup if considered with the usual multiplication and the weak operator topology. By (i) we have $0 \in S$. Define the operators $\tilde{T}(t) : C(S) \rightarrow C(S)$ by

$$(\tilde{T}(t)f)(R) := f(T(t)R), \quad f \in C(S), R \in S.$$

By Nagel (ed.) [43], Lemma B-II.3.2, $(\tilde{T}(t))_{t \geq 0}$ is a C_0 -semigroup on $C(S)$.

By Example 2.2 (c) the set $\{f(T(t) \cdot) : t \geq 0\}$ is relatively weakly compact in $C(S)$ for every $f \in C(S)$. It means that every orbit $\{\tilde{T}(t)f : t \geq 0\}$ is relatively weakly compact, and, by Lemma 2.1, $(\tilde{T}(t))_{t \geq 0}$ is a relatively weakly compact semigroup.

Denote by \tilde{P} the mean ergodic projection of $(\tilde{T}(t))_{t \geq 0}$. We have $\text{Fix}(\tilde{T}) = \bigcap_{t \geq 0} \text{Fix}(\tilde{T}(t)) = \langle \mathbf{1} \rangle$. Indeed, for $f \in \text{Fix}(\tilde{T})$ one has $f(T(t)I) = f(I)$ for all $t \geq 0$ and therefore f should be constant. Hence $\tilde{P}f$ is constant for every $f \in C(S)$. By definition of the ergodic projection

$$(1) \quad (\tilde{P}f)(0) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \tilde{T}(s)f(0) ds = f(0).$$

Thus we have

$$(2) \quad (\tilde{P}f)(R) = f(0) \cdot \mathbf{1}, \quad f \in C(S), R \in S.$$

Take now $x \in X$. By Theorem 3 and its proof in Dunford, Schwartz [12], p. 434, the weak topology on the orbit $\{T(t)x : t \geq 0\}$ is metrisable and coincides with the topology induced by some sequence $\{y_n\}_{n=1}^\infty \subset X' \setminus \{0\}$. Consider $f_{x,n} \in C(S)$ defined by

$$f_{x,n}(R) := |\langle Rx, \frac{y_n}{\|y_n\|} \rangle|, \quad R \in S,$$

and $f_x \in C(S)$ defined by

$$f_x(R) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} f_{x,n}(R), \quad R \in S.$$

By (2) we obtain

$$0 = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \tilde{T}(s)f_{x,y}(I) ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f_x(T(s)) ds.$$

Lemma 2.6 applied to the continuous and bounded function $\mathbb{R}_+ \ni t \mapsto f(T(t)I)$ yields a set $M \subset \mathbb{R}$ with density 1 such that

$$f_x(T(t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad t \in M.$$

By definition of f_x and by the fact that the weak topology on the orbit is induced by $\{y_n\}_{n=1}^\infty$ we have in particular that

$$T(t)x \xrightarrow{\sigma} 0 \quad \text{as } t \rightarrow \infty, \quad t \in M.$$

This proves (iii).

(iii) \Rightarrow (iv) follows directly from Lemma 2.6.

(iv) \Rightarrow (vii) holds by the spectral mapping theorem for the point spectrum (see Engel, Nagel [17], Thm. V.3.7).

(iv) \Leftrightarrow (v): Clearly, the semigroup $(T(t))_{t \geq 0}$ is bounded. Take $x \in X$, $y \in X'$ and let $a > 0$. By the Plancherel theorem applied to the function $t \mapsto e^{-at}\langle T(t)x, y\rangle$ we have

$$\int_{-\infty}^{\infty} |\langle R(a+is, A)x, y\rangle|^2 ds = 2\pi \int_0^{\infty} e^{-2at} |\langle T(t)x, y\rangle|^2 dt.$$

We obtain by the equivalence of Abel and Cesàro limits (see, e.g., Hardy [23], p. 136)

$$\begin{aligned} \lim_{a \rightarrow 0+} a \int_{-\infty}^{\infty} |\langle R(a+is, A)x, y\rangle|^2 ds &= 2\pi \lim_{a \rightarrow 0+} a \int_0^{\infty} e^{-2at} |\langle T(s)x, y\rangle|^2 ds \\ (3) \quad &= \pi \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\langle T(s)x, y\rangle|^2 ds. \end{aligned}$$

Note that for a bounded continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $C := \sup f(\mathbb{R}_+)$ we have

$$\left(\frac{1}{Ct} \int_0^t f^2(s) ds \right)^2 \leq \left(\frac{1}{t} \int_0^t f(s) ds \right)^2 \leq \frac{1}{t} \int_0^t f^2(s) ds,$$

which together with (3) gives the equivalence of (iv) and (v).

For the additional part of the theorem suppose X' to be separable. Then so is X , and we can take dense subsets $\{x_n \neq 0 : n \in \mathbb{N}\} \subseteq X$ and $\{y_m \neq 0 : m \in \mathbb{N}\} \subseteq X'$. Consider the functions

$$f_{n,m} : S \rightarrow \mathbb{R}, \quad f_{n,m}(R) := \left| \langle R \frac{x_n}{\|x_n\|}, \frac{y_m}{\|y_m\|} \rangle \right|, \quad n, m \in \mathbb{N},$$

which are continuous and uniformly bounded in $n, m \in \mathbb{N}$. Define the function

$$f : S \rightarrow \mathbb{R}, \quad f(R) := \sum_{n,m \in \mathbb{N}} \frac{1}{2^{n+m}} f_{n,m}(R).$$

Then clearly $f \in C(S)$. Thus, as in the proof of the implication (i') \Rightarrow (iv), i.e., using (1) we obtain

$$\frac{1}{t} \int_0^t f(T(s)I) ds \xrightarrow{t \rightarrow \infty} 0.$$

Hence, applying Lemma 2.6 to the continuous and bounded function $\mathbb{R}_+ \ni t \mapsto f(T(t)I)$ gives the existence of a set M with density 1 such that $f(T(t)) \rightarrow 0$ as $t \rightarrow \infty$, $t \in M$. In particular, $|\langle T(t)x_n, y_m \rangle| \rightarrow 0$ for all $n, m \in \mathbb{N}$ as $t \in M$, $t \rightarrow \infty$, which, together with the boundedness of $(T(t))_{t \geq 0}$, proves the implication (i') \Rightarrow (iii*). The implications (iii*) \Rightarrow (ii*) \Rightarrow (ii') are straightforward, hence the proof is complete. \square

The above theorem shows that starting from “no purely imaginary eigenvalues of the generator”, one arrives at properties like (iii) on the asymptotic behaviour of the orbits of the semigroup. This justifies the following name for this property.

Definition 2.7. We will call a relatively weakly compact C_0 -semigroup *almost weakly stable* if it satisfies one of the equivalent conditions in Theorem 2.5.

Historical remark 2.8. Theorem 2.5 and especially the implication (vii) \Rightarrow (iii) has a long history. It goes back to ergodic theory and von Neumann’s spectral mixing theorem for flows, see Halmos [22], Mixing Theorem, p. 39. This has been generalised to operators on Banach spaces by many authors, see, e.g., Nagel [42], Jones, Lin [28, 29] and Krengel [33], pp. 108–110. The implication (vii) \Rightarrow (i) appears also in Ruess, Summers [55] (see also Section 6.1). The conditions (i), (iii) and (iv) were studied by Hiai [24] also for strongly measurable semigroups. He related it to the discrete case as well. See also Kühne [34, 35].

Remark 2.9. The conditions in Theorem 2.5 are of quite different nature. Conditions (i)–(iv) as well as (ii*) and (iii*) give information on the behaviour of the semigroup $(T(t))_{t \geq 0}$, while conditions (v)–(vii) deal with the generator and its resolvent. Among them condition (vii) apparently is the simplest to verify.

Remark 2.10. It is surprising that there is a characterisation of strong stability of bounded C_0 -semigroups on Hilbert spaces which is completely analogous to the equivalence (i') \Leftrightarrow (v) in Theorem 2.5. More precisely, a bounded C_0 -semigroup $(T(t))_{t \geq 0}$ on a Hilbert space H with generator A is strongly stable if and only if

$$\lim_{a \rightarrow 0+} a \int_{-\infty}^{\infty} \|R(a + is, A)x\|^2 ds = 0$$

holds for every $x \in H$ (see Tomilov [59], pp. 108–110). We note that also the proofs of the equivalence (i') \Leftrightarrow (v) and Theorem 3.1 in [59] are analogous.

Remark 2.11. The conditions (iii) and (iii*) show that all the orbits $t \mapsto T(t)x$ converge weakly to 0 as $t \rightarrow \infty$ for t in a large subset. In general, it may happen however that this large set is not \mathbb{R}_+ , i.e., $(T(t))_{t \geq 0}$ is not weakly stable (for examples see Section 4). Here is the essential difference to strong stability: for a bounded semigroup $(T(t))_{t \geq 0}$ the convergence $\|T(t_n)x\| \rightarrow 0$ for a sequence $t_n \rightarrow \infty$ already implies $\lim_{t \rightarrow \infty} \|T(t)x\| = 0$.

3. WEAK STABILITY

As we have already noted in the introduction, a bounded C_0 -semigroup on a reflexive Banach space X induces the Jacobs-Glicksberg-de Leeuw decomposition $X = X_r \oplus X_s$, but orbits in X_s do not necessarily converge (weakly) to zero. However, in the particular case of contractive semigroups on Hilbert spaces, one can detach the subspace of all weakly stable orbits and characterise its complement.

Theorem 3.1. Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup of contractions on a Hilbert space H and define

$$W := \{x \in H : \lim_{t \rightarrow \infty} \langle T(t)x, x \rangle = 0\}.$$

Then W is a closed subspace of H , W and W^\perp are $(T(t))_{t \geq 0}$ -invariant, the restricted semigroup $(T(t)|_W)_{t \geq 0}$ is weakly stable on W and $(T(t)|_{W^\perp})_{t \geq 0}$ is unitary on W^\perp .

For the proof we refer the reader to Luo, Guo, Morgul [37], Thm. 3.18, p. 122, or see Foguel [19] for the analogous discrete case.

In the following propositions we state some immediate consequences of the above decomposition.

Proposition 3.2. *Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup of contractions on a Hilbert space H and let $x \in H$. Then the following assertions hold.*

- (i) $\lim_{t \rightarrow \infty} T(t)x = 0$ weakly if and only if $\lim_{t \rightarrow \infty} \langle T(t)x, x \rangle = 0$.
- (ii) If $(T(t))_{t \geq 0}$ is completely non-unitary, i.e., if there is no reducing subspace on which it is unitary, then $(T(t))_{t \geq 0}$ is weakly stable.

Proposition 3.3. *Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup of contractions on a Hilbert space H . Then $(T(t)|_{W^\perp})_{t \geq 0}$ has no weakly stable orbit, hence the spectral measures of its generator are non-Rajchman.*

(For the definition of Rajchman measures and a brief discussion see Example 4.2.)

We now turn to sufficient conditions for weak stability proved partly in Chill, Tomilov [9]. It is based on the behaviour of the resolvent $R(\cdot, A)$ of the generator and uses the *pseudo-spectral bound of A* (also called *abscissa of uniform boundedness of the resolvent*)

$$s_0(A) := \inf \{a \in \mathbb{R} : R(\lambda, A) \text{ is bounded on } \{\lambda : \operatorname{Re} \lambda > a\}\}.$$

Theorem 3.4. *Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup on a Banach space X with generator A satisfying $s_0(A) \leq 0$. Further, let $x \in X$ and $y \in X'$ be fixed. Consider the following assertions.*

- (a)
$$\int_0^1 \int_{-\infty}^{\infty} |\langle R^2(a + is, A)x, y \rangle| ds da < \infty.$$
- (b)
$$\lim_{a \rightarrow 0+} a \int_{-\infty}^{\infty} |\langle R^2(a + is, A)x, y \rangle| ds = 0.$$
- (c)
$$\lim_{t \rightarrow \infty} \langle T(t)x, y \rangle = 0$$

Then (a) \Rightarrow (b) \Rightarrow (c). In particular, if (a) or (b) holds for all $x \in X$ and $y \in X'$, then $(T(t))_{t \geq 0}$ is weakly stable.

Proof. First we show that (a) implies (b).

Assume that (a) holds. From the theory of Hardy spaces we know that the function $f : (0, 1) \mapsto \mathbb{R}_+$ defined by

$$f(a) := \int_{-\infty}^{\infty} |\langle R^2(a + is, A)x, y \rangle| ds$$

is monotone decreasing for $a > 0$ (see Rosenblum, Rovnyak [49] for the theory of Hardy spaces). Assume now that (b) is not true. Then there exists a monotonic decreasing null sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$(4) \quad a_n f(a_n) \geq c$$

holds for some $c > 0$ and all $n \in N$.

Take now any $n, m \in \mathbb{N}$ such that $a_n \leq \frac{a_m}{2}$. By (4) and monotonicity of f we have

$$\int_{a_n}^{a_m} f(a) da \geq \sum_{k=n}^{m-1} (a_k - a_{k+1}) f(a_k) \geq \frac{c}{a_n} (a_m - a_n) = c \left(\frac{a_m}{a_n} - 1 \right) \geq c$$

holds. This contradicts (a) and the implication (a) \Rightarrow (b) is proved.

It remains to show that (b) implies (c).

By (b) we have for every $a > 0$

$$\int_{-\infty}^{\infty} |\langle R^2(a + is, A)x, y \rangle| ds < \infty.$$

Moreover, condition $s_0(A) \leq 0$ implies that the function $\lambda \mapsto \langle R^2(\lambda, A)x, y \rangle$ is bounded on every half-plane $\{\lambda : \operatorname{Re} \lambda \geq a\}$. Therefore, it belongs to the Hardy space $H^1(\{\lambda : \operatorname{Re} \lambda > a\})$ and

$$\int_{-\infty}^{\infty} |\langle R^2(a + is, A)x, y \rangle| ds < \infty$$

holds for all $a > 0$. This allows us to represent the semigroup as the inverse Laplace transform for all $a > \max\{0, \omega_0(T)\}$, where $\omega_0(T)$ is the growth bound of $(T(t))_{t \geq 0}$. Indeed, from e.g. Kaashoek, Verduyn Lunel [30] or Kaiser, Weis [31] it follows that

$$(5) \quad \langle T(t)x, y \rangle = \frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{(a+is)t} \langle R^2(a + is, A)x, y \rangle ds.$$

A standard application of Cauchy's theorem extends the validity of (5) to all $a > 0$. We now take $t = \frac{1}{a}$ to obtain

$$|\langle T(t)x, y \rangle| \leq a \int_{-\infty}^{\infty} |\langle R^2(a + is, A)x, y \rangle| ds \rightarrow 0$$

as $a \rightarrow 0+$, so $t = \frac{1}{a} \rightarrow \infty$. □

The implication (b) \Rightarrow (c) is stated in Chill, Tomilov [9]. They also show that the strong analogues of (a) and (b) both imply strong stability of the semigroup. Note that the relation (a) \Rightarrow (b) is also valid for the strong case by the same arguments.

We conclude this section with the following remarkable fact about weak stability. By Theorem 2.5 one has almost weak stability under quite general assumptions. As we will see in the next section, almost weak stability does not imply weak stability and, moreover, the difference between these two concepts is fundamental (see Theorem 4.6). In particular, this means that weak convergence of the semigroup to zero along some sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \rightarrow \infty$ in general does not imply weak stability. However, once the sequence $\{t_n\}_{n=1}^{\infty}$ is relatively dense, i.e., there exists a number $\ell > 0$ such that every sub-interval of \mathbb{R}_+ of length ℓ intersects $\{t_n : n \in \mathbb{N}\}$ (see Bart, Goldberg [3] for the terminology), one *does obtain* weak stability.

Theorem 3.5. *Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup on a Banach space X . Suppose that*

$\lim_{n \rightarrow \infty} T(t_n) = 0$ weakly for some relatively dense sequence $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$. Then $(T(t))_{t \geq 0}$ is weakly stable.

Proof. Without loss of generality assume that $\{t_n\}_{n=1}^\infty$ is monotone increasing and set $\ell := \sup_{n \in \mathbb{N}} (t_{n+1} - t_n)$, which is finite by assumption. Since every C_0 -semigroup is bounded on compact time intervals, and $(T(t_n))_{n \in \mathbb{N}}$ is weakly converging, hence bounded, we obtain that the semigroup $(T(t))_{t \geq 0}$ is bounded.

Fix $x \in X$, $y \in X'$. For $t \in [t_n, t_{n+1}]$ we have

$$\langle T(t)x, y \rangle = \langle T(t-t_n)x, T'(t_n)y \rangle,$$

where $(T'(t))_{t \geq 0}$ is the adjoint semigroup. We note that by assumption $T'(t_n)y \rightarrow 0$ in the weak*-topology.

Further, the set $K_x := \{T(s)x : 0 \leq s \leq \ell\}$ is compact in X and $T(t-t_n)x \in K_x$ for every $n \in \mathbb{N}$. Since pointwise convergence is equivalent to the uniform convergence on compact sets (see, e.g., [17], Prop. A.3), we see that $\langle T(t)x, y \rangle \rightarrow 0$. \square

Note that taking $t_n := n$ in Theorem 3.5 we obtain that $(T(t))_{t \geq 0}$ is weakly stable if and only if $T(n) \rightarrow 0$ weakly as $n \rightarrow \infty$, $n \in \mathbb{N}$. This gives a connection between weak stability of discrete and continuous semigroups.

4. EXAMPLES

In this section we discuss concrete and abstract examples of almost weakly but not weakly stable semigroups. Finally, we present recent results showing that weakly and almost weakly stable semigroups even have different Baire category in spaces of unitary and isometric operators on Hilbert spaces.

The first example indicates how one can construct almost weakly but not weakly stable semigroups using dynamical systems arising in ergodic theory.

Example 4.1. A measurable measure-preserving semiflow $(\varphi_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{M}, \mu)$ is called *strongly mixing* if $\lim_{t \rightarrow \infty} \mu(\varphi_t^{-1}(A) \cap B) = \mu(A)\mu(B)$ for any two measurable sets $A, B \in \mathcal{M}$. The semiflow $(\varphi_t)_{t \geq 0}$ is called *weakly mixing* if for all $A, B \in \mathcal{M}$ we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\mu(\varphi_s^{-1}(A) \cap B) - \mu(A)\mu(B)| ds = 0.$$

These concepts play an essential role in ergodic theory, and we refer to the monographs Cornfeld, Fomin, Sinai [11], Krengel [33], Petersen [48], or Halmos [22] for further information. Clearly, strong mixing implies weak mixing, but the converse implication does not hold in general. However, examples of weakly but not strongly mixing semiflows are not easy to construct; see Lind [36] for an example and Petersen [48], p. 209 for a method of constructing such semiflows.

The semiflow $(\varphi_t)_{t \geq 0}$ on $(\Omega, \mathcal{M}, \mu)$ induces a semigroup of isometries $(T(t))_{t \geq 0}$ on each of the Banach spaces $X = L^p(\Omega, \mu)$ ($1 \leq p < \infty$) by defining

$$(T(t)f)(\omega) := f(\varphi_t(\omega)), \quad \omega \in \Omega, f \in L^p(\Omega, \mu).$$

This semigroup is strongly continuous (see Krengel [33], §1.6, Thm. 6.13) and relatively weakly compact by virtue of Example 2.2 (b) with $u = \mathbf{1}$. It is well-known (see, e.g., Halmos [22], pp. 37–38) that

$$(\varphi_t)_{t \geq 0} \text{ is strongly mixing} \iff \lim_{t \rightarrow \infty} \langle T(t)f, g \rangle = \langle Pf, g \rangle \text{ for all } f \in X, g \in X',$$

and

$$(\varphi_t)_{t \geq 0} \text{ is weakly mixing} \iff \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\langle T(s)f, g \rangle - \langle Pf, g \rangle| ds = 0 \text{ for all } f \in X, g \in X',$$

where P is the projection onto $\text{Fix}(T)$ given by $Pf := \int_{\Omega} f d\mu \cdot \mathbf{1}$ for all $f \in X$. Note that in both cases $\text{Fix}(T) = \langle \mathbf{1} \rangle$ holds.

Take now any semiflow $(\varphi_t)_{t \geq 0}$ which is weakly but not strongly mixing. Observe that $X = X_0 \oplus \langle \mathbf{1} \rangle$, where

$$X_0 := \left\{ f \in X : \int_{\Omega} f d\mu = 0 \right\}$$

is closed and $(T(t))_{t \geq 0}$ -invariant. We denote the restriction of $(T(t))_{t \geq 0}$ to X_0 by $(T_0(t))_{t \geq 0}$ and its generator by A_0 . The semigroup $(T_0(t))_{t \geq 0}$ is still relatively weakly compact and, since $P\sigma(A) \cap i\mathbb{R} = \emptyset$, it is almost weakly stable. On the other hand, $(T_0(t))_{t \geq 0}$ is not weakly stable since $(\varphi_t)_{t \geq 0}$ is not strongly mixing.

We can also look at this example from a different perspective. If $(\varphi_t)_{t \in \mathbb{R}}$ is even a measure preserving flow, it induces a C_0 -group $(T(t))_{t \in \mathbb{R}}$ of unitary operators on the Hilbert space $L^2(\Omega, \mu)$. Hence we can apply the spectral theorem and obtain for each $x \in H$ a measure ν_x on \mathbb{R} such that

$$\langle T(t)x, x \rangle = \int_{\mathbb{R}} e^{itr} d\nu_x(r) \quad \text{for all } t \geq 0.$$

Thus $\langle T(t)x, x \rangle$ becomes the Fourier transform of the measure ν_x . In the next example we classify these measures according to the behaviour of their Fourier transform at infinity.

Example 4.2. Let us consider the Hilbert space $H = L^2(\mathbb{R}, \mu)$, where μ is a finite positive Borel measure, and the operator A on H is the multiplication operator

$$Af(r) = irf(r), \quad r \in \mathbb{R},$$

on its maximal domain. Then A generates the unitary group $(T(t)f)(r) := e^{itr} f(r)$. Since Hilbert spaces are reflexive, $(T(t))_{t \geq 0}$ is relatively weakly compact by Example 2.2 (a).

Clearly, $\sigma(A) \subseteq i\mathbb{R}$ and $ir \in i\mathbb{R}$ is an eigenvalue of A if and only if $\mu(\{r\}) > 0$. Hence, if $\mu(\{r\}) = 0$ for all $r \in \mathbb{R}$, then A has no eigenvalues and the Jacobs-Glicksberg-de Leeuw decomposition yields that $(T(t))_{t \geq 0}$ is almost weakly stable.

For $f, g \in H$ we have $\langle T(t)f, g \rangle = \int_{\mathbb{R}} e^{itr} f(r) \overline{g}(r) d\mu$. In particular, by taking $f = g = \mathbf{1}$ we obtain $\langle T(t)\mathbf{1}, \mathbf{1} \rangle = \int_{\mathbb{R}} e^{itr} d\mu = \mathcal{F}\mu(t)$, the Fourier transform of μ . On the other hand, $\lim_{t \rightarrow \infty} \mathcal{F}\mu(t) = 0$ implies $\lim_{t \rightarrow \infty} \langle T(t)f, g \rangle = 0$ for all $f, g \in H$, therefore

$$(T(t))_{t \geq 0} \text{ is weakly stable} \iff \lim_{t \rightarrow \infty} \mathcal{F}\mu(t) = 0.$$

Note that since for unitary groups weak stability as $t \rightarrow \infty$ coincides with weak stability as $t \rightarrow -\infty$, the property above is equivalent to

$$\lim_{|t| \rightarrow \infty} \mathcal{F}\mu(t) = 0.$$

In harmonic analysis, this property of the measure μ got its own name. Indeed, μ is called *Rajchman* if its Fourier transform vanishes at infinity. We refer to Lyons [38, 39] for a brief historical overview on these measures and their properties.

We note that absolutely continuous measures are always Rajchman by the Riemann-Lebesgue lemma and all Rajchman measures are continuous by Wiener's theorem. However, there are continuous measures which are not Rajchman and Rajchman measures which are not absolutely continuous (see Lyons [39]). It is now a consequence of the considerations above that each continuous non-Rajchman measure gives rise to an almost weakly but not weakly stable unitary group. In Engel, Nagel [17], p. 316 an example of a unitary group even with bounded generator is given, for which the corresponding spectral measures are not Rajchman.

Next, we give an example of a positive semigroup on a Banach lattice which is almost weakly stable but not weakly stable.

Example 4.3. As in Nagel (ed.) [43], p. 206, we start from a flow on $\mathbb{C} \setminus \{0\}$ with the following properties:

- 1) The orbits starting in z with $|z| \neq 1$ spiral towards the unit circle Γ ;
- 2) 1 is the fixed point of φ and $\Gamma \setminus \{1\}$ is a homoclinic orbit, i.e., $\lim_{t \rightarrow -\infty} \varphi_t(z) = \lim_{t \rightarrow \infty} \varphi_t(z) = 1$ for every $z \in \Gamma$.

A concrete example comes from the differential equation in polar coordinates $(r, \omega) = (r(t), \omega(t))$:

$$\begin{cases} \dot{r} = 1 - r, \\ \dot{\omega} = 1 + (r^2 - 2r \cos \omega). \end{cases}$$

Take $x_0 \in \mathbb{C}$ with $0 < |x_0| < 1$ and denote by $S_{x_0} := \{\varphi_t(x_0) : t \geq 0\}$ the orbit starting from x_0 . Then $S := S_{x_0} \cup \Gamma$ is compact for the usual topology of \mathbb{C} .

We define a multiplication on S as follows. For $x = \varphi_t(x_0)$ and $y = \varphi_s(x_0)$ we put

$$xy := \varphi_{t+s}(x_0).$$

For $x \in \Gamma$, $x = \lim_{n \rightarrow \infty} x_n$, $x_n = \varphi_{t_n}(x_0) \in S_{x_0}$ and $y = \varphi_s(x_0) \in S_{x_0}$, we define $xy = yx := \lim_{n \rightarrow \infty} x_n y$. Note that by $|x_n y - \varphi_s(x)| = |\varphi_s(x_n) - \varphi_s(x)| \leq C|x_n - x| \xrightarrow{n \rightarrow \infty} 0$ the definition is correct and satisfies

$$xy = \varphi_s(x).$$

For $x, y \in \Gamma$ we define $xy := 1$. This multiplication on S is separately continuous and makes S a semi-topological semigroup (see Engel, Nagel [17], Sec. V.2).

Consider now the Banach space $X := C(S)$. By Example 2.2 (c) the set

$$\{f(s \cdot) : s \in S\} \subset C(S)$$

is relatively weakly compact for every $f \in C(S)$. By definition of the multiplication on S this implies that

$$\{f(\varphi_t(\cdot)) : t \geq 0\}$$

is relatively weakly compact in $C(S)$. Consider the semigroup induced by the flow, i.e.,

$$(T(t)f)(x) := f(\varphi_t(x)), \quad f \in C(S), x \in S.$$

By the above, each orbit $\{T(t)f : t \geq 0\}$ is relatively weakly compact in $C(S)$ and hence, by Lemma 2.1, $(T(t))_{t \geq 0}$ is weakly compact. Note that the strong continuity of $(T(t))_{t \geq 0}$ follows, as shown in Nagel (ed.) [43], Lemma B-II.3.2, from the separate continuity of the flow. Furthermore, the semigroup $(T(t))_{t \geq 0}$ is isometric.

Next, we take $X_0 := \{f \in C(S) : f(1) = 0\}$ and identify it with the Banach lattice $C_0(S \setminus \{1\})$. Then both subspaces in the decomposition $C(S) = X_0 \oplus \langle \mathbf{1} \rangle$ are invariant under $(T(t))_{t \geq 0}$.

Denote by $(T_0(t))_{t \geq 0}$ the restricted semigroup to X_0 and by A_0 its generator. The semigroup $(T_0(t))_{t \geq 0}$ is still relatively weakly compact.

Since $\text{Fix}(T_0) := \bigcap_{t \geq 0} \text{Fix}(T_0(t)) = \{0\}$, we have that $0 \notin P\sigma(A_0)$. Moreover, $P\sigma(A_0) \cap i\mathbb{R} = \emptyset$ holds, which implies by the Jacobs-Glicksberg-de Leeuw theorem that $(T_0(t))_{t \geq 0}$ is almost weakly stable.

To see that $(T_0(t))_{t \geq 0}$ is not weakly stable it is enough to consider $\delta_{x_0} \in X'_0$. Since

$$\langle T_0(t)f, \delta_{x_0} \rangle = f(\varphi(t, x_0)), \quad f \in X_0,$$

$f(\Gamma)$ always belongs to the closure of $\{\langle T_0(t)f, \delta_{x_0} \rangle : t \geq 0\}$ and hence the semigroup $(T_0(t))_{t \geq 0}$ can not be weakly stable.

Let us summarise the above as follows.

Proposition 4.4. *There exist a locally compact space Ω and a positive, relatively weakly compact C_0 -semigroup of isometries on $C_0(\Omega)$ which is almost weakly but not weakly stable.*

This enables us to answer a question of Emelyanov [16] in negative. Consider the discrete semigroup $(T(n))_{n \in \mathbb{N}} = (T(1)^n)_{n \in \mathbb{N}}$ from Proposition 4.4. By a result of Jones and Lin [29], we know that 0 belongs to the weak closure of each of the orbits. Whereas Theorem 3.5 shows that this semigroup is not weakly stable. The semigroup is positive and isometric on the Banach lattice $C_0(\Omega)$.

Moreover, Proposition 4.4 becomes particularly interesting in view of the following results; for details and discussion see Chill, Tomilov [10].

Theorem 4.5 (Groh, Neubrander [21], Thm. 3.2; Chill, Tomilov [10], Thm. 7.7). *For a bounded, positive, mean ergodic C_0 -semigroup $(T(t))_{t \geq 0}$ on a Banach lattice X with generator $(A, D(A))$, the following assertions hold.*

- (i) *If $X \cong L^1(\Omega, \mu)$, then $P\sigma(A) \cap i\mathbb{R} = \emptyset$ is equivalent to the strong stability of $(T(t))_{t \geq 0}$.*
- (ii) *If $X \cong C(K)$, K compact, then $P\sigma(A) \cap i\mathbb{R} = \emptyset$ is equivalent to the uniform exponential stability of $(T(t))_{t \geq 0}$.*

Example 4.3 shows that we can not drop the assumption on the existence of a unit element in X in Theorem 4.5 (ii).

At the end of this section, we show that the examples presented above represent the general situation. Indeed, typical isometric semigroups and typical unitary groups are almost weakly but not weakly stable in the following sense.

Theorem 4.6. *Let H be a separable infinite-dimensional Hilbert space and let \mathcal{I} denote the set of all isometric C_0 -semigroups on H . Then the set of all weakly stable isometric semigroups is of first category and the set of all almost weakly stable isometric semigroups is residual in \mathcal{I} with respect to an appropriate topology making \mathcal{I} to a completely metrisable space. The analogous statement holds for unitary groups.*

For the proof and precise description of the appropriate topologies on the spaces of all unitary and all isometric semigroups see Eisner, Serény [15], and [14] for the case of unitary, isometric and contractive operators. These results are the operator theoretic counterpart of classical category theorems of Halmos and Rokhlin from ergodic theory, see Halmos [22], pp. 77–80.

5. INDIVIDUAL STABILITY AND LOCAL RESOLVENT

In this section, we restrict our attention to single orbits and present results implying

$$\lim_{t \rightarrow \infty} \langle T(t)x, y \rangle = 0 \quad \text{for some given } x \text{ and } y.$$

The tool will be the *bounded local resolvent* $R(\lambda)x_0$ which exists by definition if the function $\rho(A) \ni \lambda \mapsto R(\lambda, A)x_0$ admits a bounded, holomorphic extension $R(\lambda)x_0$ to the whole right half-plane $\{\lambda : \operatorname{Re} \lambda > 0\}$. This we assume in the following.

Clearly, if we suppose that for *all* $x_0 \in X$ the local resolvent $R(\lambda)x_0$ is bounded, analyticity and the principle of uniform boundedness yield the boundedness of the operator resolvent $R(\lambda, A)$ on $\{\lambda : \operatorname{Re} \lambda > 0\}$, hence uniform exponential stability for semigroups on Hilbert spaces and (at least) strong stability for semigroups on reflexive Banach spaces (see the theorems of Gearhart and Arendt, Batty, Lyubich, Vũ, e.g., in Engel, Nagel [17], Thm. V.1.11 and V.2.21). The reasonable questions therefore address the individual stability of a single orbit in terms of the local resolvent of one single element $x_0 \in X$.

Without any differentiability or boundedness assumption on the semigroup it is necessary to do some initial smoothing on x_0 in order to have stability in any sense. However, if the Banach space X has some nice geometric properties, even strong stability can be derived. Huang and van Neerven [26] proved that if the Banach space is B -convex or has the analytic Radon-Nikodým property, then the existence of a bounded local resolvent $R(\lambda)x_0$ on $\{\lambda : \operatorname{Re} \lambda > 0\}$ already implies *strong convergence* $T(t)R(\mu, A)^\alpha x_0 \rightarrow 0$ as $t \rightarrow +\infty$ for any $\alpha > 1$. (Here μ is greater than the growth bound $\omega_0(A)$, thus $R(\mu, A)$ is a sectorial operator admitting fractional powers.) Actually, if X has Fourier type $p > 1$, then we can take $\alpha > 1/p$, see [26] (the assumption that $\alpha > 1/p$ is shown to be optimal by Wrobel [61]), and if the semigroup is eventually differentiable and $p = 2$, then no smoothing is needed, i.e., $\alpha \geq 0$ is allowed, see [25].

In general, without any additional assumptions on the space or on the regularity of the semigroup one can only deduce weak individual stability. The following result is due to Huang and van Neerven [26].

Theorem 5.1. *Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup on a Banach space X with generator A . Let $x_0 \in X$ and suppose that the function $\lambda \mapsto R(\lambda, A)x_0$ has a bounded holomorphic extension to $\{\lambda : \operatorname{Re} \lambda > 0\}$. Then*

- a) $\lim_{t \rightarrow \infty} \langle T(t)x_0, y \rangle = 0$ for all $y \in D((A')^2)$;
- b) $\lim_{t \rightarrow \infty} T(t)x_0 = 0$ weakly, if additionally the semigroup $(T(t))_{t \geq 0}$ is uniformly bounded.

Tauberian theorems are among the primary tools to deduce information on the asymptotic behaviour of the semigroup from properties of the resolvent, and they have extensively been used to obtain strong and weak stability. We refer the reader to the monograph Arendt, Batty, Hieber, Neubrander [2] and also to Chill [8]. As illustration we include here a proof for part b) which uses Ingham's Tauberian theorem.

Theorem 5.2 (Ingham). *Let $f : \mathbb{R}_+ \rightarrow \mathbb{C}$ be bounded and uniformly continuous and suppose that the Laplace transform \hat{f} of f has a locally integrable boundary function on the imaginary axis (that is, there exists $h \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C})$ such that $\lim_{a \rightarrow 0+} \hat{f}(a + i \cdot) = h$ in the distributional sense). Then $\lim_{t \rightarrow \infty} f(t) = 0$.*

For proofs and a detailed treatment see [2], Sect. 4 and [8].

Proof of part b). By assumption the operator resolvent $R(\lambda, A)x_0$ and the local resolvent $R(\lambda)x_0$ coincide on the right halfplane. So for a fix $y \in X'$, on the right half-plane $\lambda \mapsto \langle R(\lambda)x_0, y \rangle$ is the Laplace transform of the function $t \mapsto \langle T(t)x_0, y \rangle$. Since $(T(t))_{t \geq 0}$ is uniformly bounded, the weak orbit $t \mapsto \langle T(t)x_0, y \rangle$ is bounded and uniformly continuous, so we can apply Ingham's theorem to obtain $\lim_{t \rightarrow \infty} \langle T(t)x_0, y \rangle = 0$. \square

For the proof of Theorem 5.1 part a) one could also use Ingham's Theorem, and check the assumptions of the theorem by following the lines of Batty, Chill and van Neerven in [5]. Actually in [5] a powerful functional calculus method is developed, which among other yields the proof of the more general Theorem 5.3 below. To prove part a) we nevertheless choose a different, fairly elementary way (see Eisner, Farkas [13]).

Proof of part a). By $\lambda \mapsto R(\lambda)x_0$ we denote the holomorphic continuation of $\lambda \mapsto R(\lambda, A)x_0$ to the half-plane $\{\lambda : \operatorname{Re} \lambda > 0\}$. The uniqueness theorem for holomorphic functions and the resolvent identity imply

$$\begin{aligned} R(\delta + is)x_0 &= R(a + is, A)x_0 + (a - \delta)R(a + is, A)R(\delta + is)x_0 \\ &= R(a + is, A)x_0 + (a - \delta)R^2(a + is, A)x_0 + (a - \delta)^2R^2(a + is, A)R(\delta + is)x_0. \end{aligned}$$

For all $y \in D(A'^2)$ we have

$$\begin{aligned} 2\pi e^{-\delta t} \langle T(t)x_0, y \rangle &= \int_{-\infty}^{\infty} e^{ist} \langle R(\delta + is)x_0, y \rangle ds = \int_{-\infty}^{\infty} e^{ist} \langle R(a + is, A)x_0, y \rangle ds \\ &\quad + (a - \delta) \int_{-\infty}^{\infty} e^{ist} \langle R^2(a + is, A)x_0, y \rangle ds + (a - \delta)^2 \int_{-\infty}^{\infty} e^{ist} \langle R^2(a + is, A)R(\delta + is)x_0, y \rangle ds. \end{aligned}$$

Indeed, for $a > \omega_0(T)$ the first equality follows from representing the semigroup as the inverse Laplace transform of the resolvent; subsequently, the Cauchy theorem implies this representation for all $a > 0$. The functions $f_\delta(s) := \langle R^2(a + is, A)R(\delta + is)x_0, y \rangle$ form a relatively compact subset of $L^1(\mathbb{R})$, because

$$\begin{aligned} |f_\delta(s)| &= |\langle R^2(a + is, A)R(\delta + is)x_0, y \rangle| = \\ &= |\langle R(\delta + is)x_0, R^2(a + is, A')y \rangle| \leq M \|R^2(a + is, A')y\|, \end{aligned}$$

and the function on the right hand side lies in $L^1(\mathbb{R})$, so the family f_δ is uniformly integrable (and bounded), thus relatively compact. By compactness we find a sequence $\delta_n \rightarrow 0$ such that $\lim_{n \rightarrow \infty} f_{\delta_n} = f$ in $L^1(\mathbb{R})$. By substituting δ_n in the above equality and letting $n \rightarrow \infty$ we obtain

$$\begin{aligned} 2\pi \langle T(t)x_0, y \rangle &= \int_{-\infty}^{\infty} e^{ist} \langle R(a + is, A)x_0, y \rangle ds \\ &\quad + a \int_{-\infty}^{\infty} e^{ist} \langle R^2(a + is, A)x_0, y \rangle ds + a^2 \int_{-\infty}^{\infty} e^{ist} f(s) ds = I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

It is easy to deal with the last term I_3 . The function f lies in $L^1(\mathbb{R})$, so by the Riemann-Lebesgue Lemma its Fourier transform vanishes at infinity, i.e., $\lim_{t \rightarrow \infty} I_3(t) = 0$. Since $y \in D((A')^2)$, we

can integrate by parts in I_1 to obtain

$$I_1(t) = \int_{-\infty}^{\infty} e^{ist} \langle x_0, R(a+is, A')y \rangle ds = \frac{1}{t} \int_{-\infty}^{\infty} e^{ist} \langle x_0, R^2(a+is, A')y \rangle ds.$$

The last integral is absolutely convergent, because using the resolvent identity for $R(\lambda, A')$, one can show that for $y \in D(A'^2)$ and $a > \omega_0(T)$ fixed, $\|R^2(a+is, A')y\| = O((a^2 + s^2)^{-1})$ holds. Hence

$$|I_1(t)| \leq \frac{1}{t} \int_{-\infty}^{\infty} \|x_0\| \cdot \|R^2(a+is, A')y\| ds \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Concerning I_2 we observe that $\langle x_0, R^2(a+i\cdot, A')y \rangle \in L^1(\mathbb{R})$, so once again by the Riemann-Lebesgue Lemma we have

$$I_2(t) = a \int_{-\infty}^{\infty} e^{ist} \langle x_0, R^2(a+is, A')y \rangle ds \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

and the proof is complete. \square

Actually, Huang and van Neerven proved the following more general theorem.

Theorem 5.3 (Huang, van Neerven [26]). *Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup on a Banach space X with generator A . For $x_0 \in X$ assume that the bounded local resolvent exists. Then $\lim_{t \rightarrow \infty} T(t)(\lambda_0 - A)^{-\beta}x_0 = 0$ weakly for all $\beta > 1$ and $\lambda_0 > \omega_0(T)$.*

Under a special positivity condition one can take $\beta = 1$ in Theorem 5.3.

Theorem 5.4 (van Neerven [47]). *Suppose that X is an ordered Banach space with weakly closed normal cone C . If for some $x_0 \in X$ the function $\lambda \mapsto R(\lambda, A)x_0$ has a bounded holomorphic extension to $\{\lambda : \operatorname{Re} \lambda > 0\}$ and $T(t)x_0 \in C$ for all sufficiently large t , then $\lim_{t \rightarrow \infty} T(t)R(\mu, A)x_0 = 0$ weakly for all $\mu \in \rho(A)$.*

The above eventual positivity assumption cannot be omitted. Indeed, van Neerven [47] proved that the existence of a bounded local resolvent $R(\lambda)x_0$ in general implies $\|T(t)R(\mu, A)x_0\| = O(1+t)$, and Batty [4] showed this to be optimal, whereas weak convergence of $T(t)R(\mu, A)x_0$ to zero would imply $\|T(t)R(\mu, A)x_0\| = O(1)$.

6. COMMENTS AND OPEN QUESTIONS

In this closing section, we collect some noteworthy open questions. We also touch upon two further areas connected to the topic of this survey: weakly almost periodic functions and the cogenerator of contraction semigroups on Hilbert spaces.

We start by listing some problems arising from this paper.

Question 6.1. In Remark 2.10 we have seen that in Hilbert spaces strong stability is fully characterised by an integrability condition of the resolvent of the generator. This raises the question if the same is true for weak stability, i.e., for example, does the converse of Theorem 3.4 hold in Hilbert spaces? Note the difference between Theorem 3.4 b) and Theorem 2.5 (v), the latter being equivalent to almost weak stability.

Question 6.2. Understanding weak stability of unitary groups is of special importance, but a satisfactory description is still lacking. We mention here that not even the boundedness of the generator of the group would make the problem easier (see Engel and Nagel [17], p. 316).

Question 6.3. By Theorem 3.5 it suffices to understand weak stability of discrete semigroups. However, a description of those sequences $\{t_n\}_{n=1}^\infty \subseteq \mathbb{R}$ for which $\lim_{n \rightarrow \infty} T(t_n) = 0$ weakly implies $\lim_{t \rightarrow \infty} T(t) = 0$ weakly is desirable.

6.1. Weakly almost periodic functions. In this part, we sketch very briefly some results of Ruess and Summers ([50]–[55]) in connection with weak asymptotic behaviour of operator semigroups. We select only those aspects of their theory that are directly related to weak stability of orbits. However, to illustrate the merit of their approach, we shall consider the more general setting of *almost orbits*, too. In the sequel, we assume that $(T(t))_{t \geq 0}$ is a uniformly bounded C_0 -semigroup with generator A . A function $u : \mathbb{R}_+ \rightarrow X$ is called an *almost orbit* of the semigroup $(T(t))_{t \geq 0}$ if

$$\lim_{t \rightarrow \infty} \sup_{h \in \mathbb{R}_+} \|u(t+h) - T(h)u(t)\| = 0 \quad \text{holds.}$$

The use of this notion is explained for instance by the fact that a solution of an inhomogeneous abstract Cauchy problem is an almost orbit of the corresponding semigroup.

We shall need the following definition; see [50] and the references therein for details and historical remarks (cf also Bart, Goldberg [3]).

Definition 6.4. A bounded, continuous function $f \in C_b(\mathbb{R}_+, X)$ is called *asymptotically almost periodic* if its translates $H(f) := \{f(\cdot + h) : h \in \mathbb{R}_+\}$ form a relatively compact set in $C_b(\mathbb{R}_+, X)$ for the sup-norm. A function $f \in C_b(\mathbb{R}_+, X)$ is called *Eberlein-weakly almost periodic*, if $H(f)$ is relatively weakly compact in $C_b(\mathbb{R}_+, X)$. The spaces of such functions are denoted by $AAP(\mathbb{R}_+, X)$ and $W(\mathbb{R}_+, X)$ respectively, and $W_0(\mathbb{R}_+, X)$ is the vector subspace of functions $f \in W(\mathbb{R}_+, X)$ such that the weak closure of $H(f)$ contains 0. A function $f \in C_b(\mathbb{R}_+, X)$ is *weakly asymptotically almost periodic* if for all $y \in X'$ the function $y \circ f : \mathbb{R}_+ \rightarrow \mathbb{C}$ belongs to $AAP(\mathbb{R}_+, \mathbb{C})$.

Clearly, an asymptotically almost periodic function is Eberlein-weakly almost periodic and weakly asymptotically almost periodic, while the converse implications fail in general.

It is not surprising that the function $t \mapsto T(t)x$ is Eberlein-weakly almost periodic if and only if the orbit $\{T(t)x : t \geq 0\}$ is relatively weakly compact (this is also true for almost orbits, see [53], Sect. 4.5). The structure theory for Eberlein-weakly almost periodic functions developed by Ruess and Summers thus gives a slightly finer description of relatively weakly compact semigroups than the Glicksberg-Jacobs-de Leeuw decomposition. Using this approach Ruess and Summers proved the following (for orbits this is basically a corollary of the Jacobs-Glicksberg-de Leeuw decomposition, see Theorem 2.5).

Theorem 6.5 (Ruess, Summers [53]). *Let $(T(t))_{t \geq 0}$ be a uniformly bounded C_0 -semigroup with generator A on the reflexive Banach space X . Then the following are equivalent:*

- a) $P\sigma(A) \cap i\mathbb{R} = \emptyset$.
- b) Every almost orbit of $(T(t))_{t \geq 0}$ belongs to $W_0(\mathbb{R}_+, X)$.
- c) Every orbit of $(T(t))_{t \geq 0}$ belongs to $W_0(\mathbb{R}_+, X)$.
- d) $\liminf_{t \rightarrow \infty} |\langle T(t)x, y \rangle| = 0$ for each $x \in X$, $y \in X'$.

As for weak stability, asymptotic almost periodicity is of great use.

Theorem 6.6 (Ruess, Summers [55]). *Let u be a weakly asymptotically almost periodic almost orbit of $(T(t))_{t \geq 0}$ and suppose that its range $\{u(t) : t \geq 0\}$ is relatively weakly compact. Then the following hold:*

- a) *If $P\sigma(A) \cap i\mathbb{R} = \emptyset$, then $\lim_{t \rightarrow \infty} u(t) = 0$ weakly.*
- b) *If $P\sigma(A) \cap i\mathbb{R} = \{0\}$, then $u(t)$ has a weak limit, which is a fixed point of $(T(t))_{t \geq 0}$.*

For a detailed treatment, as well as for connections to ergodic theory and further generalisations we refer to [50, 51, 52, 53, 54, 55] (see also Arendt, Batty, Hieber, Neubrander [2], Section 5.4).

6.2. Cogenerator of contractive semigroups. Let $(T(t))_{t \geq 0}$ be a contractive C_0 -semigroup on a Hilbert space H . The *cogenerator* of $(T(t))_{t \geq 0}$ is defined as the (negative) Cayley-transform of the infinitesimal generator A of $(T(t))_{t \geq 0}$, i.e.,

$$G := -(I + A)(I - A)^{-1} = I - 2R(1, A).$$

It is easy to see that the cogenerator is a contraction, see Sz.-Nagy, Foiaş [58], Sections III.8–9 for details. Using a functional calculus the semigroup can be directly recovered and, moreover, many important properties of the semigroup can be read off from its cogenerator. Namely, the semigroup consists of normal, self-adjoint, isometric or unitary operators if and only if the cogenerator is normal, self-adjoint, isometric or unitary, respectively. Also the asymptotic behaviour of the semigroup can be characterised with the help of the cogenerator.

Theorem 6.7. *Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup of contractions and G its cogenerator. Then*

$$\lim_{t \rightarrow \infty} \|T(t)x\| = \lim_{n \rightarrow \infty} \|G^n x\|.$$

In particular, the semigroup is strongly stable if and only if G is strongly stable.

Motivated by this we ask the following.

Question 6.8. Is the analogue of Theorem 6.7 true for weak stability? More precisely, is there a connection between the weak stability of a contractive semigroup and the weak stability of its cogenerator?

Note that the function $z \mapsto -\frac{1+z}{1-z}$ maps the imaginary axis onto the unit circle, so by the spectral mapping theorem for the point spectrum (see Engel, Nagel [17], Thm. IV.3.7), we have that

$$P\sigma(A) \cap i\mathbb{R} = \emptyset \quad \text{if and only if} \quad P\sigma(G) \cap \{z : |z| = 1\} = \emptyset.$$

Hence by a result of Jones and Lin [29] we obtain the following.

Proposition 6.9. *A contraction semigroup $(T(t))_{t \geq 0}$ on a Hilbert space H is almost weakly stable if and only if its cogenerator G is “almost weakly stable”, i.e., when 0 belongs to the weak closure of each orbit $\{G^n x : n \in \mathbb{N}\}$, $x \in H$.*

This again connects the asymptotic behaviour of $(T(t))_{t \geq 0}$ to the behaviour of the powers of a single operator, thus allowing the application of results by, e.g., Jones and Lin [28, 29].

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